Application of *Maple* to Lagrangian Mechanics

Frank Wang
Department of Physics, Columbia University,
538 West 120th Street, New York, NY 10027; fw@phys.columbia.edu

The Lagrangian formulation of mechanics has great advantages in practical use. Mathematics involved in this subject is the calculus of variations, which involves finding a function that minimizes an integral. Calculations of this type require finding the derivative of a function with respect to another function. *Maple* provides many procedures to support such differentiation, and there are many integrated procedures to derive the equations of motion directly. Indeed, the latest version *Maple* 8 has a new package `VariationalCalculus` that incorporates all necessary calculations. These procedures are tremendously convenient in practical use, but their shortcomings are that one needs to invoke external procedures for *Maple* to perform such calculation, and that perhaps a novice user tends not to comprehend fully the underlying physics. We develop a method in *Maple* to solve problems of the calculus of variations. In our approach, we use only basic commands such as `subs`, `diff`, and `dsolve`, without programming or invoking an external library. Although our method is a pedagogic approach that might involve more manual steps, it is a direct attack on this problem, and practically all problems in classical mechanics can be solved once the Lagrangian is found. In most real physical problems, there is no analytic solution to differential equations. We particularly emphasize forming plots based on a numeric method; by varying physical parameters and initial conditions and observing the graphic outputs, one can develop an intuition about the underlying physics. In this presentation, to illustrate our method we begin with a simple problem, finding the shortest connection between two points in a plane. We then proceed to problems of a double pendulum and of a heavy symmetric top. Our final example is in general relativity, to find the trajectory of a particle near a black hole, which corresponds to nothing other than the shortest connection between two points in a curved space.

1 Calculus of Variations

We consider first a problem in one-dimensional form. A function $f(y, y', x)$ is defined on a path $y = y(x)$ between two values x_a and x_b, in which y' is the derivative of y with respect to x,

$$y' \equiv \frac{dy}{dx},$$

We seek a particular path $y(x)$ such that the line integral J

$$J = \int_{x_a}^{x_b} f(y, y', x)dx$$

has extremal value. Solution of this problem requires that the following condition be satisfied:

$$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0}. \tag{1}$$

This equation is called the Euler-Lagrange equation. A derivation of this equation can be found in many books, such as *Classical Mechanics* by Goldstein (1980) [1], p. 37ff. We utilize repeatedly this equation in this work.

To demonstrate how to use the Euler-Lagrange equation, we begin with a simple problem. What is the plane curve of minimum length connecting two given points?

The element of arc length in a plane, which is an infinitesimal separation between two points, is

$$ds = \sqrt{dx^2 + dy^2}$$

The total length of any curve going between points $\mathcal{A}$ and $\mathcal{B}$ is hence

$$J = \int_{\mathcal{A}}^{\mathcal{B}} ds = \int_{x_a}^{x_b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

We identify f as

$$f = \sqrt{1 + y'^2}$$

Substituting f into the Euler-Lagrange equation and solving the differential equation, we find the desired function.

To employ the Euler-Lagrange equation, we must find the derivative of f with respect to $y(x)$ and $y'(x)$, both of which are functions. We can substitute them with two symbols, such as *var1* and *var2*. This simple substitution makes it possible for differentiation to proceed without invoking an external procedure. Furthermore, it introduces an important concept that we treat $y(x)$ and $y'(x)$ as two separate variables.

Defining the integrand as a function of $y(x)$ and $y'(x)$,

```
> f := sqrt(1 + diff(y(x),x)^2);
```

$$f := \sqrt{1 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}$$

and substituting $y(x)$ and $y'(x)$ as two symbols, namely *var1* and *var2* respectively,

```
> f1 := subs({y(x)=var1, diff(y(x),x)=var2}, f);
```

$$\mathit{f1} := \sqrt{1 + \mathit{var2}^2}$$

We differentiate f with respect to those symbols.

```
> Eq1 := diff(f1, var2);
```

$$\mathit{Eq1} := \frac{\mathit{var2}}{\sqrt{1 + \mathit{var2}^2}}$$

```
> Eq2 := diff(f1, var1);
```

$$\mathit{Eq2} := 0$$

We substitute *var1* and *var2* back to their original assignments,

```
> Eq3 := subs({var1=y(x), var2=diff(y(x),x)}, Eq1);
```

$$\mathit{Eq3} := \frac{\frac{\partial}{\partial x}\,\mathrm{y}(x)}{\sqrt{1 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}}$$

```
> Eq4 := subs({var1=y(x), var2=diff(y(x),x)}, Eq2);
```

$$\mathit{Eq4} := 0$$

We can find the derivative with respect to the independent variable x,

```
> Eq5 := diff(Eq3, x);
```

$$\mathit{Eq5} := -\frac{(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))}{(1 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)^{(3/2)}} + \frac{\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)}{\sqrt{1 + (\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}}$$

The Euler-Lagrange equation states that

```
> Eq6 := Eq5 - Eq4 = 0;
```

$$Eq6 := -\frac{(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2\,(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x))}{(1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2)^{(3/2)}} + \frac{\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)}{\sqrt{1+(\frac{\partial}{\partial x}\,\mathrm{y}(x))^2}} = 0$$

Using command `dsolve`, we solve the differential equation.

```
> Eq7 := dsolve(Eq6, y(x));
```

$$Eq7 := \mathrm{y}(x) = \_C1,\ \mathrm{y}(x) = \_C1\ x + \_C2$$

The nontrivial solution is hence

$$y = c_1 x + c_2$$

which is, as expected, a straight line.

Although these substitutions seem tedious, they are readily accomplished with standard editing operations such as "copy" and "paste" in *Maple*.

2 Lagrange's Equations

Hamilton's principle in integral form states that the motion of a system from time t_1 to t_2 is such that the line integral

$$I = \int_{t_1}^{t_2} L\,dt$$

takes the least value for the correct path of motion. The integrand L is called the Lagrangian. For a classical mechanical system, a Lagrangian is the difference of kinetic energy and potential energy,

$$L = T - V. \tag{2}$$

We can directly apply the Euler-Lagrange equation to this Lagrangian, so to find the equations of motion. For a system with freedom in many degrees, the equation of motion is derived as

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}, \tag{3}$$

in which q_i are generalized coordinates. We discuss generalized coordinates through examples in the next two sections.

3 Double Pendulum

The problem of a double pendulum is a classic example of the Lagrangian formulation, see [1] p. 14. The positions of two masses that constitute a double pendulum can be specified as x_1, y_1 and x_2, y_2, but they are not independent of each other. We use the respective angles to the vertical, θ_1 and θ_2, as generalized coordinates; they are related to Cartesian coordinates through these transformation equations:

$$x_1 = l_1 \sin\theta_1, \quad y_1 = l_1 \cos\theta_1$$

and

$$x_2 = x_1 + l_2 \sin\theta_2, \quad y_2 = y_1 + l_2 \cos\theta_2$$

The kinetic energy is

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2),$$

and the potential energy is

$$V = -m_1 g y_1 - m_2 g y_2.$$

We obtain the Lagrangian

$$L = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)] + m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2). \quad (4)$$

Employing the Euler-Lagrange equation, we have for θ_1,

$$(m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\cos(\theta_1-\theta_2)\ddot{\theta}_2 + m_2 l_1 l_2\sin(\theta_1-\theta_2)\dot{\theta_2}^2 + (m_1+m_2)g l_1\sin\theta_1 = 0 \quad (5)$$

and for θ_2,

$$m_2 l_1 l_2\cos(\theta_1-\theta_2)\ddot{\theta}_1 + m_2 l_2^2\ddot{\theta}_2 - m_2 l_1 l_2\sin(\theta_1-\theta_2)\dot{\theta_1}^2 + m_2 g l_2\sin\theta_2 = 0 \quad (6)$$

The algebraic manipulations to simplify the kinetic energy and to employ the Euler-Lagrange equation are tedious if they are carried out manually, but using *Maple* eliminates pointless labor. Furthermore, these two coupled differential equations admit no analytic solution. One can apply an approximation of a small angle to linearize the equations, so to obtain the normal modes. We can, however, directly apply the command `dsolve` with option `numeric` in *Maple* to solve the equations numerically. To implement a numerical approach, we must provide four initial conditions for two differential equations of second order: they are $\theta_1(0)$, $\theta_2(0)$, $\dot{\theta_1}(0)$, and $\dot{\theta}_2(0)$.

```
> x1 := l1*sin(theta1(t));
```

$$x1 := l1\sin(\theta 1(t))$$

```
> y1 := l1*cos(theta1(t));
```

$$y1 := l1\cos(\theta 1(t))$$

```
> x2 := x1 + l2*sin(theta2(t));
```

$$x2 := l1\sin(\theta 1(t)) + l2\sin(\theta 2(t))$$

```
> y2 := y1 + l2*cos(theta2(t));
```

$$y2 := l1\cos(\theta 1(t)) + l2\cos(\theta 2(t))$$

Maple can undertake the rearrangement, which would be long and tedious with manual operation; to verify this statement, one can replace the colon with a semicolon and observe the *Maple* outcome.

```
> T := 1/2*m1*(diff(x1,t)^2 + diff(y1,t)^2) + 1/2*m2*(diff(x2,t)^2

> + diff(y2,t)^2):
> T := combine(T);
```

$$T := \frac{1}{2}\,m1\,l1^2\,(\tfrac{\partial}{\partial t}\theta 1(t))^2 + \frac{1}{2}\,m2\,l1^2\,(\tfrac{\partial}{\partial t}\theta 1(t))^2$$
$$+\,m2\,l1\,(\tfrac{\partial}{\partial t}\theta 1(t))\,l2\,(\tfrac{\partial}{\partial t}\theta 2(t))\cos(\theta 1(t) - \theta 2(t)) + \frac{1}{2}\,m2\,l2^2\,(\tfrac{\partial}{\partial t}\theta 2(t))^2$$

The potential energy is

```
> V := -m1*g*y1 - m2*g*y2;
```

$$V := -m1\,g\,l1\cos(\theta 1(t)) - m2\,g\,(l1\cos(\theta 1(t)) + l2\cos(\theta 2(t)))$$

We obtain the Lagrangian for the double pendulum system.

```
> L := T - V;
```

$$L := \frac{1}{2}\, m1\; l1^2\, (\tfrac{\partial}{\partial t}\,\theta 1(t))^2 + \frac{1}{2}\, m2\; l1^2\, (\tfrac{\partial}{\partial t}\,\theta 1(t))^2$$
$$+ m2\; l1\; (\tfrac{\partial}{\partial t}\,\theta 1(t))\; l2\; (\tfrac{\partial}{\partial t}\,\theta 2(t)) \cos(\theta 1(t) - \theta 2(t)) + \frac{1}{2}\, m2\; l2^2\, (\tfrac{\partial}{\partial t}\,\theta 2(t))^2$$
$$+ m1\; g\; l1 \cos(\theta 1(t)) + m2\; g\, (l1 \cos(\theta 1(t)) + l2 \cos(\theta 2(t)))$$

There are two degrees of freedom; the Lagrangian is specified by θ_1, θ_2, and their temporal derivatives. We again assign symbols such as *var1*, *var2* etc.

```
> L1 := subs({theta1(t)=var1, diff(theta1(t), t)=var2, theta2(t)=var3,
> diff(theta2(t), t)=var4}, L):
```

After these substitutions, we employ our standard method. For coordinate θ_1,

```
> Eq11 := diff(L1, var2);
```

$$Eq11 := m1\; l1^2\; var2 + m2\; l1^2\; var2 + m2\; l1\; l2\; var4 \cos(var1 - var3)$$

```
> Eq12 := diff(L1, var1);
```

$$Eq12 := -m2\; l1\; var2\; l2\; var4 \sin(var1 - var3) - m1\; g\; l1 \sin(var1) - m2\; g\; l1 \sin(var1)$$

```
> Eq13 := subs({var1=theta1(t), var2=diff(theta1(t), t),
> var3=theta2(t),
> var4=diff(theta2(t), t)}, Eq11);
```

$$Eq13 := m1\; l1^2\, (\tfrac{\partial}{\partial t}\,\theta 1(t)) + m2\; l1^2\, (\tfrac{\partial}{\partial t}\,\theta 1(t)) + \cos(\theta 1(t) - \theta 2(t))\, (\tfrac{\partial}{\partial t}\,\theta 2(t))\; l1\; l2\; m2$$

```
> Eq14 := subs({var1=theta1(t), var2=diff(theta1(t), t),
> var3=theta2(t),
> var4=diff(theta2(t), t)}, Eq12);
```

$$Eq14 := -m2\; l1\; (\tfrac{\partial}{\partial t}\,\theta 1(t))\; l2\; (\tfrac{\partial}{\partial t}\,\theta 2(t)) \sin(\theta 1(t) - \theta 2(t)) - m1\; g\; l1 \sin(\theta 1(t))$$
$$- m2\; g\; l1 \sin(\theta 1(t))$$

```
> Eq15 := diff(Eq13, t);
```

$$Eq15 := m1\; l1^2\, (\tfrac{\partial^2}{\partial t^2}\,\theta 1(t)) + m2\; l1^2\, (\tfrac{\partial^2}{\partial t^2}\,\theta 1(t))$$
$$- \sin(\theta 1(t) - \theta 2(t))\, ((\tfrac{\partial}{\partial t}\,\theta 1(t)) - (\tfrac{\partial}{\partial t}\,\theta 2(t)))\, (\tfrac{\partial}{\partial t}\,\theta 2(t))\; l1\; l2\; m2$$
$$+ \cos(\theta 1(t) - \theta 2(t))\, (\tfrac{\partial^2}{\partial t^2}\,\theta 2(t))\; l1\; l2\; m2$$

```
> Eq16 := Eq15 - Eq14 = 0;
```

$$Eq16 := m1\; l1^2\, (\tfrac{\partial^2}{\partial t^2}\,\theta 1(t)) + m2\; l1^2\, (\tfrac{\partial^2}{\partial t^2}\,\theta 1(t))$$
$$- \sin(\theta 1(t) - \theta 2(t))\, ((\tfrac{\partial}{\partial t}\,\theta 1(t)) - (\tfrac{\partial}{\partial t}\,\theta 2(t)))\, (\tfrac{\partial}{\partial t}\,\theta 2(t))\; l1\; l2\; m2$$
$$+ \cos(\theta 1(t) - \theta 2(t))\, (\tfrac{\partial^2}{\partial t^2}\,\theta 2(t))\; l1\; l2\; m2$$
$$+ m2\; l1\; (\tfrac{\partial}{\partial t}\,\theta 1(t))\; l2\; (\tfrac{\partial}{\partial t}\,\theta 2(t)) \sin(\theta 1(t) - \theta 2(t)) + m1\; g\; l1 \sin(\theta 1(t))$$
$$+ m2\; g\; l1 \sin(\theta 1(t)) = 0$$

```
> Eq17 := collect(Eq16, diff);
```

$$Eq17 := (m1\; l1^2 + l1^2\; m2)\, (\tfrac{\partial^2}{\partial t^2}\,\theta 1(t)) + \cos(\theta 1(t) - \theta 2(t))\, (\tfrac{\partial^2}{\partial t^2}\,\theta 2(t))\; l1\; l2\; m2$$
$$+ \sin(\theta 1(t) - \theta 2(t))\; l1\; l2\; m2\; (\tfrac{\partial}{\partial t}\,\theta 2(t))^2 + m2\; g\; l1 \sin(\theta 1(t)) + m1\; g\; l1 \sin(\theta 1(t)) =$$
$$0$$

and for coordinate θ_2 (suppressing output from intermediate steps),

```
> Eq21 := diff(L1, var4):
```

```
> Eq22 := diff(L1, var3):
```

```
> Eq23 := subs({var1=theta1(t), var2=diff(theta1(t), t),
> var3=theta2(t),
> var4=diff(theta2(t), t)}, Eq21):
```

```
> Eq24 := subs({var1=theta1(t), var2=diff(theta1(t), t),
> var3=theta2(t),
> var4=diff(theta2(t), t)}, Eq22):
```

```
> Eq25 := diff(Eq23, t):
```

```
> Eq26 := Eq25 - Eq24 = 0:
```

```
> Eq27 := collect(Eq26, diff);
```

$$Eq27 := m2\ l1\ (\frac{\partial^2}{\partial t^2}\,\theta1(t))\ l2\cos(\theta1(t)-\theta2(t)) + m2\ l2^2\,(\frac{\partial^2}{\partial t^2}\,\theta2(t)) + m2\ g\ l2\sin(\theta2(t)) \\ -\sin(\theta1(t)-\theta2(t))\,(\frac{\partial}{\partial t}\,\theta1(t))^2\ l1\ l2\ m2 = 0$$

Because these two coupled differential equations admit no analytic solution, we directly supply numerical values and initial conditions to solve these equations numerically.

```
> m1 := 0.05;  m2 := 0.05;  l1 := 0.5;  l2 := 0.5;  g := 9.8;
```

$$m1 := .05$$
$$m2 := .05$$
$$l1 := .5$$
$$l2 := .5$$
$$g := 9.8$$

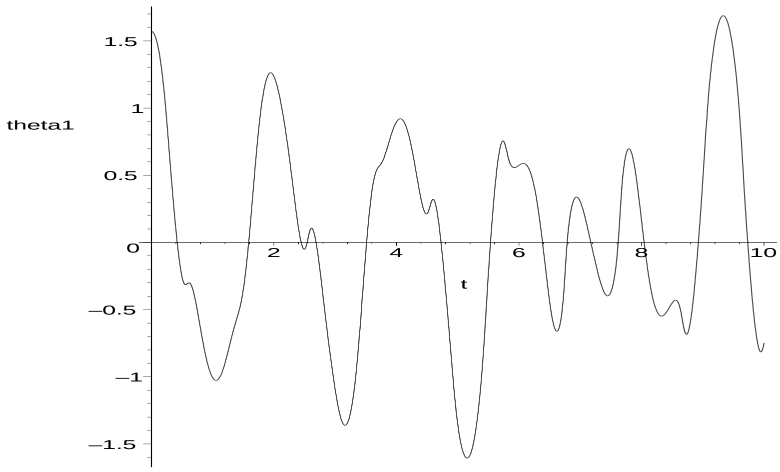
```
> ini := theta1(0)=Pi/2.0, D(theta1)(0)=0, theta2(0)=Pi/4.0,
> D(theta2)(0)=0;          # Choose your values.
```

$$ini := \theta1(0) = .5000000000\,\pi,\ \mathrm{D}(\theta1)(0) = 0,\ \theta2(0) = .2500000000\,\pi,\ \mathrm{D}(\theta2)(0) = 0$$

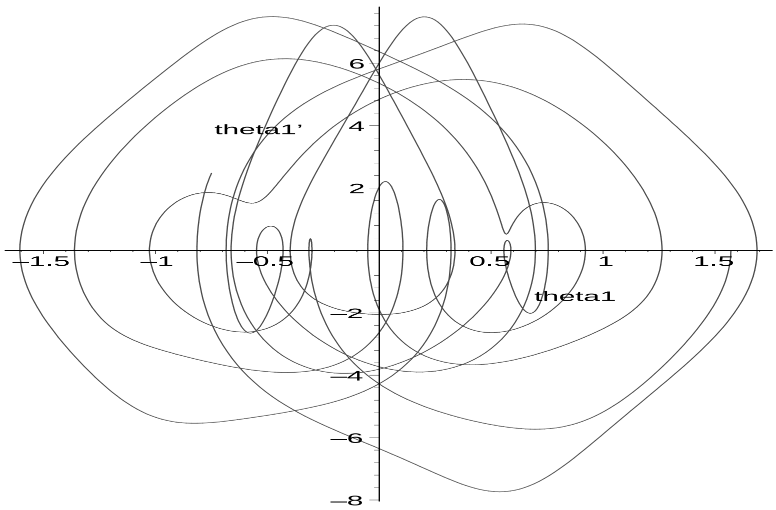
```
> Eq75 := dsolve({Eq17, Eq27, ini}, {theta1(t), theta2(t)},
> numeric, output=listprocedure):
```

```
> with(plots):    with(plottools):
Warning, the name changecoords has been redefined
Warning, the name arrow has been redefined
```

```
> odeplot(Eq75, [t, theta1(t)], 0..10, numpoints=200);
```




```
> odeplot(Eq75, [theta1(t), diff(theta1(t),t)], 0..10, numpoints=200);
```



We form an animation by displaying plot frames in sequence.

```
> noffm := 100:

> divs := 10:
> for i from 0 to noffm do

> x1 := l1*sin(rhs(Eq75[2](i/divs))):

> y1 := -l1*cos(rhs(Eq75[2](i/divs))):

> x2 := x1 + l2*sin(rhs(Eq75[4](i/divs))):

> y2 := y1 - l2*cos(rhs(Eq75[4](i/divs))):

> rod[i] := curve([[0,0], [x1,y1], [x2,y2]]):

> ms1[i] := disk([x1,y1], 0.02, color=red):

> ms2[i] := disk([x2,y2], 0.02, color=blue):

> anima[i] := display({rod[i], ms1[i], ms2[i]}):

> end do:
```

```
> display([seq(anima[i], i=0..noffm)], insequence=true,
> scaling=constrained, axes=none);
```

---

4 Motion of a Heavy Symmetric Top

We discuss next a symmetric top with one point fixed under the influence of gravity, discussed at length in Goldstein [1], p. 213ff. Let the axis of symmetry be taken as z fixed in the body; about this axis the moment of inertia is I_3. By symmetry $I_1 = I_2$. The kinetic energy is

$$T = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2, \tag{7}$$

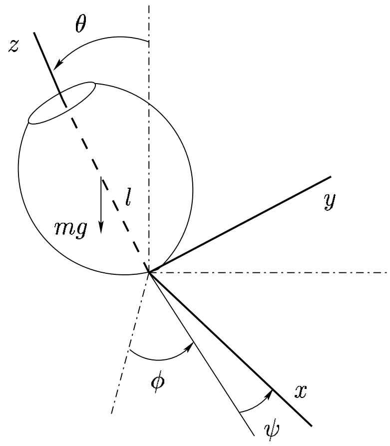


Figure 1: Euler's angles specifying the orientation of a symmetrical top.

It is convenient to specify the conformation of the top in three Euler's angles: θ gives the inclination of axis z from the vertical, ϕ measures the azimuth of the top about the vertical direction, and ψ is the angle of rotation of the top about its own z axis; see Figure 1. This notation conforms to that of Goldstein [1], p. 214; in this source one can also find a detailed derivation of kinetic energy in terms of Euler's angles. We state the result, which is

$$T = \frac{I_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\psi} + \dot{\phi} \cos \theta)^2. \tag{8}$$

The potential energy is

$$V = mgl \cos \theta, \tag{9}$$

so the Lagrangian is

$$L = \frac{I_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\psi} + \dot{\phi} \cos \theta)^2 - mgl \cos \theta. \tag{10}$$

Because ϕ and ψ do not appear explicitly in the Lagrangian, the corresponding generalized momenta are constant in time; see [1] p. 54ff. Because $\partial L / \partial \psi = 0$, the Euler-Lagrange equation becomes

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\psi}} \equiv \frac{dp_\psi}{dt} = 0,$$

which signifies that $p_\psi =$ constant,

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) \equiv M_z, \tag{11}$$

and similarly for p_ϕ,

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_3(\dot\psi + \dot\phi\cos\theta)\cos\theta + I_1\dot\phi\sin^2\theta \equiv M_{z'}. \tag{12}$$

For the θ coordinate, the Euler-Lagrange equation yields

$$I_1\ddot\theta - (I_1 - I_3)\dot\phi^2\sin\theta\cos\theta + I_3\dot\phi\dot\psi\sin\theta - mgl\sin\theta = 0. \tag{13}$$

With initial conditions, θ_0, $\dot\theta_0$, ϕ_0, $\dot\phi_0$, and generally known $\dot\psi$, we can solve the above differential equations.

We generally rearrange coupled differential equations to diminish the number of variables in one equation as much as possible. For $\dot\phi$ and $\dot\psi$, we can write them in terms of θ,

$$\dot\phi = \frac{M_{z'} - M_z\cos\theta}{I_1\sin^2\theta}, \tag{14}$$

$$\dot\psi = \frac{M_z}{I_3} - \frac{M_{z'} - M_z\cos\theta}{I_1\sin^2\theta}\cos\theta. \tag{15}$$

We can then obtain an equation for θ alone:

$$I_1\ddot\theta - \frac{(M_{z'} - M_z\cos\theta)^2\cos\theta}{I_1\sin^3\theta} + \frac{(M_{z'} - M_z\cos\theta)M_z}{I_1\sin\theta} - mgl\sin\theta = 0. \tag{16}$$

As M_z and $M_{z'}$ are constant, we can evaluate them from the provided initial conditions,

$$M_z = I_3(\dot\psi + \dot\phi_0\cos\theta_0), \tag{17}$$

and

$$M_{z'} = I_3(\dot\psi + \dot\phi_0\cos\theta_0)\cos\theta_0 + I_1\dot\phi_0\sin^2\theta_0. \tag{18}$$

Once the θ equation is solved, we can integrate the ϕ and ψ equations.

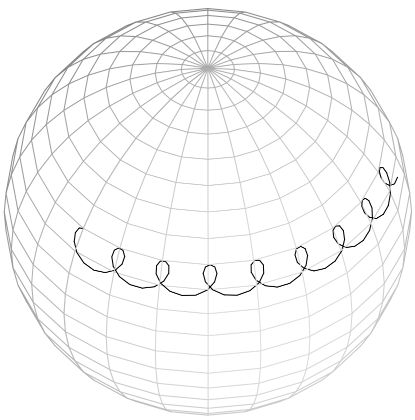


Figure 2: Locus of the figure axis.

We avoid listing lengthy code, but refer interested readers to retrieve it from the *Maple* Application Center at http://www.mapleapps.com/categories/science/physics/html/heavytop.html. With specified initial conditions, we can solve the differential equations numerically. A plot of locus of the figure axis is shown in Figure 2, which depicts the motion of the top on tracing the curve of the intersection of the figure axis on a sphere of unit radius about the fixed point.

5 Trajectory near a Black Hole

John Wheeler summarized general relativity with a statement "geometry tells matter how to move, matter tells geometry how to curve"[2], p. 130. This statement signifies that the presence of mass curves space, and the trajectory of a massive particle in a curved space is the shortest connection between two points. Above we solve the problem of finding the shortest connection between two points in a plane; for curved space, exactly the same mathematics of applying the Euler-Lagrange equation are relevant: the differential equations therefrom are called geodesic equations.

One solution of general relativity corresponding to the exterior of a spherically symmetric gravitational source M is the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \dfrac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{19}$$

which indicates a curved space.

Similar to the example of finding the shortest plane curve connecting two points, we need to minimize the integral

$$\delta \int ds = 0. \tag{20}$$

One can prove that, if at one instant θ is $\pi/2$, it retains this constant value. By considering the trajectory on the equatorial plane, we then reduce the metric as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \dfrac{2M}{r}} + r^2 d\phi^2 \tag{21}$$

We let s be the independent variable, and $t(s)$, $r(s)$, and $\phi(s)$ be dependent ones, and identify f as

$$f = \frac{1}{2}\left[-\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{ds}\right)^2 + \frac{1}{1 - \dfrac{2M}{r}}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2\right]$$

Because f does not depend explicitly on t and ϕ, we introduce two constants a and h,

$$\frac{\partial f}{\partial\left(\frac{dt}{ds}\right)} = -a, \quad \left(c^2 - \frac{2GM}{r}\right)\frac{dt}{ds} = a,$$

and

$$\frac{\partial f}{\partial\left(\frac{d\phi}{ds}\right)} = h, \quad r^2\frac{d\phi}{ds} = h.$$

For the r coordinate,

$$\frac{d}{ds}\frac{\partial f}{\partial\left(\frac{dr}{ds}\right)} - \frac{\partial f}{\partial r} = 0, \quad \frac{1}{1 - \frac{2M}{r}}\frac{d^2r}{ds^2} - \frac{1}{\left(1 - \frac{2M}{r}\right)^2}\frac{M}{r^2}\left(\frac{dr}{ds}\right)^2 + \frac{M}{r^2}\left(\frac{dt}{ds}\right)^2 - r\left(\frac{d\phi}{ds}\right)^2 = 0.$$

To rearrange the above three equations, we first isolate dt/ds and $d\phi/ds$,

$$\frac{d\phi}{ds} = \frac{h}{r^2}, \tag{22}$$

$$\frac{dt}{ds} = \frac{a}{1 - \frac{2M}{r}} \tag{23}$$

so to obtain an equation of r alone,

$$\frac{d^2r}{ds^2} - \left(1 - \frac{2M}{r}\right)\frac{h^2}{r^3} + \left[a^2 - \left(\frac{dr}{ds}\right)^2\right]\frac{1}{1-\frac{2M}{r}}\frac{M}{r^2} = 0. \tag{24}$$

We can find numerical solution to this differential equation.

Defining f,

```
> f := 1/2*(-(1 - 2*M/r(s))*diff(t(s), s)^2
> + 1/(1 - 2*M/r(s))*diff(r(s), s)^2 +
> r(s)^2*diff(phi(s),s)^2);
```

$$f := -\frac{1}{2}\,(1 - \frac{2\,M}{\mathrm{r}(s)})\,(\tfrac{\partial}{\partial s}\,\mathrm{t}(s))^2 + \frac{\frac{1}{2}\,(\frac{\partial}{\partial s}\,\mathrm{r}(s))^2}{1 - \frac{2\,M}{\mathrm{r}(s)}} + \frac{1}{2}\,\mathrm{r}(s)^2\,(\tfrac{\partial}{\partial s}\,\phi(s))^2$$

and making the substitutions,

```
> f1 := subs({t(s)=var1, diff(t(s),s)=var2, r(s)=var3,
> diff(r(s),s)=var4,
> phi(s)=var5, diff(phi(s),s)=var6}, f):
```

There is no explicit $t(s)$ term; therefore there is a constant, which we denote as $-a$.

```
> Eq11 := diff(f1, var2):
> Eq12 := diff(f1, var1):
> Eq13 := subs({var1=t(s), var2=diff(t(s),s), var3=r(s), var5=phi(s),
> var4=diff(r(s),s), var6=diff(phi(s),s)}, Eq11):
> Eq14 := Eq13 = -a;
```

$$\mathit{Eq14} := -(1 - \frac{2\,M}{\mathrm{r}(s)})\,(\tfrac{\partial}{\partial s}\,\mathrm{t}(s)) = -a$$

Applying the Euler-Lagrange equation to the r coordinate yields

```
> Eq21 := diff(f1, var4):
> Eq22 := diff(f1, var3):
> Eq23 := subs({var1=t(s), var2=diff(t(s),s), var3=r(s), var5=phi(s),
> var4=diff(r(s),s), var6=diff(phi(s),s)}, Eq21):
> Eq24 := subs({var1=t(s), var2=diff(t(s),s), var3=r(s), var5=phi(s),
> var4=diff(r(s),s), var6=diff(phi(s),s)}, Eq22):
> Eq25 := diff(Eq23, s):
> Eq26 := Eq25 - Eq24 = 0;
```

$$\mathit{Eq26} := -\frac{(\frac{\partial}{\partial s}\,\mathrm{r}(s))^2\,M}{(1 - \frac{2\,M}{\mathrm{r}(s)})^2\,\mathrm{r}(s)^2} + \frac{\frac{\partial^2}{\partial s^2}\,\mathrm{r}(s)}{1 - \frac{2\,M}{\mathrm{r}(s)}} + \frac{M\,(\frac{\partial}{\partial s}\,\mathrm{t}(s))^2}{\mathrm{r}(s)^2} - \mathrm{r}(s)\,(\tfrac{\partial}{\partial s}\,\phi(s))^2 = 0$$

There is also no explicit $\phi(t)$ term, so we denote another constant h.

```
> Eq31 := diff(f1, var6):
> Eq32 := diff(f1, var5):
```

```
> Eq33 := subs({var1=t(s), var2=diff(t(s),s), var3=r(s), var5=phi(s),
> var4=diff(r(s),s), var6=diff(phi(s),s)}, Eq31):
> Eq34 := Eq33 = h;
```

$$Eq34 := \mathrm{r}(s)^2\,(\tfrac{\partial}{\partial s}\,\phi(s)) = h$$

With some rearrangements we decouple the equations,

```
> Eq53 := isolate(Eq14, diff(t(s),s));
```

$$Eq53 := \tfrac{\partial}{\partial s}\,\mathrm{t}(s) = -\frac{a}{-1 + \dfrac{2\,M}{\mathrm{r}(s)}}$$

```
> Eq54 := isolate(Eq34, diff(phi(s),s));
```

$$Eq54 := \tfrac{\partial}{\partial s}\,\phi(s) = \frac{h}{\mathrm{r}(s)^2}$$

```
> Eq55 := subs({Eq53, Eq54}, Eq26);
```

$$Eq55 := -\frac{(\frac{\partial}{\partial s}\,\mathrm{r}(s))^2\,M}{(1 - \dfrac{2\,M}{\mathrm{r}(s)})^2\,\mathrm{r}(s)^2} + \frac{\frac{\partial^2}{\partial s^2}\,\mathrm{r}(s)}{1 - \dfrac{2\,M}{\mathrm{r}(s)}} + \frac{M\,a^2}{\mathrm{r}(s)^2\,(-1 + \dfrac{2\,M}{\mathrm{r}(s)})^2} - \frac{h^2}{\mathrm{r}(s)^3} = 0$$

and solve these differential equations numerically.

```
> M := 1;
```

$$M := 1$$

```
> Eq77 := r(0) = 11; #(12 for capture) (13 for unbound)
```

$$Eq77 := \mathrm{r}(0) = 11$$

```
> Eq78 := D(r)(0) = 0;
```

$$Eq78 := \mathrm{D}(r)(0) = 0$$

```
> Eq79 := phi(0) = 0;
```

$$Eq79 := \phi(0) = 0$$

```
> Eq80 := D(phi)(0) = 0.0295; #(0.0248 for capture) (0.0330 for
> unbound)
```

$$Eq80 := \mathrm{D}(\phi)(0) = .0295$$

The constants a and h that we assign above are related to energy and angular momentum; they can be evaluated from initial conditions.

```
> h := rhs(Eq77)^2*rhs(Eq80);
```

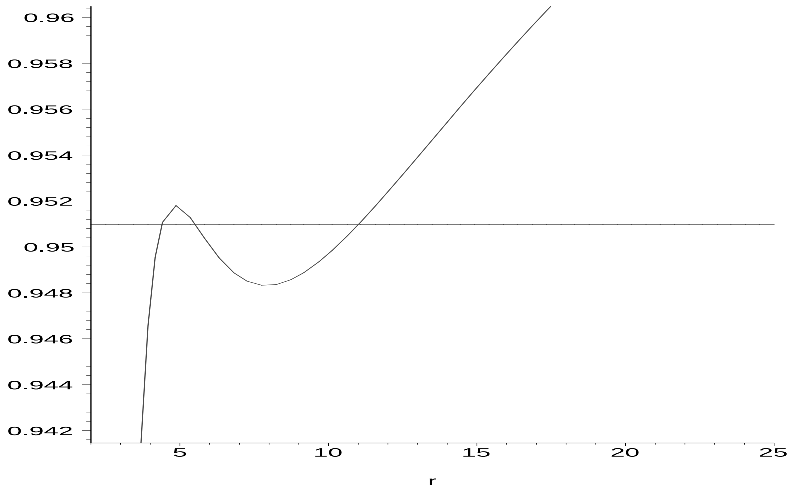
$$h := 3.5695$$

```
> a := sqrt((1 - 2*M/rhs(Eq77))*(1 + h^2/rhs(Eq77)^2));
```

$$a := .9509661236$$

We make a plot of "effective potential", which is unrelated to our approach; we list it so that the reader can compare with a conventional textbook such as in [2], p. 660.

```
> Vsq := sqrt((1 - 2*M/r)*(1 + h^2/r^2)):
> plot([Vsq, a], r=2..25, a-0.01*a..a+0.01*a, color=[red,blue]);
```




```
> ini := Eq77, Eq78, Eq79;
```

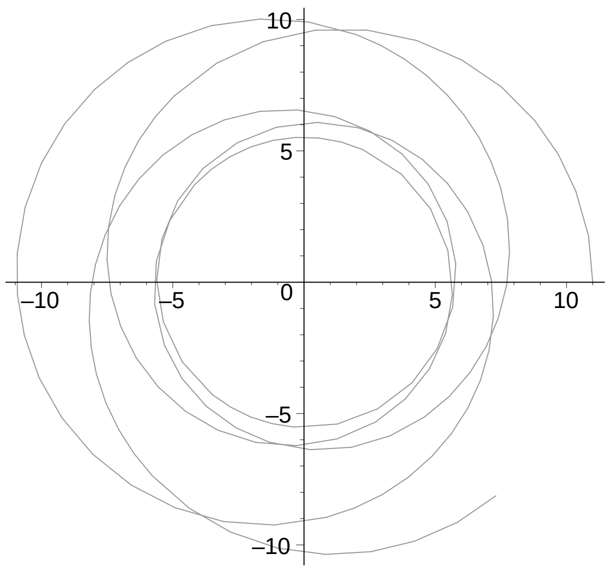
$$ini := \mathrm{r}(0) = 11,\ \mathrm{D}(r)(0) = 0,\ \phi(0) = 0$$

```
> Eq91 := dsolve({Eq54, Eq55, ini}, {r(s), phi(s)}, numeric,

> output=listprocedure):
> with(plots):
```

```
Warning, the name changecoords has been redefined
```

We plot the trajectory of a massive particle in a strong gravitational field.

```
> polarplot([rhs(Eq91(s)[3]), rhs(Eq91(s)[2]), s=0..500],

> scaling=constrained, color=coral);
```



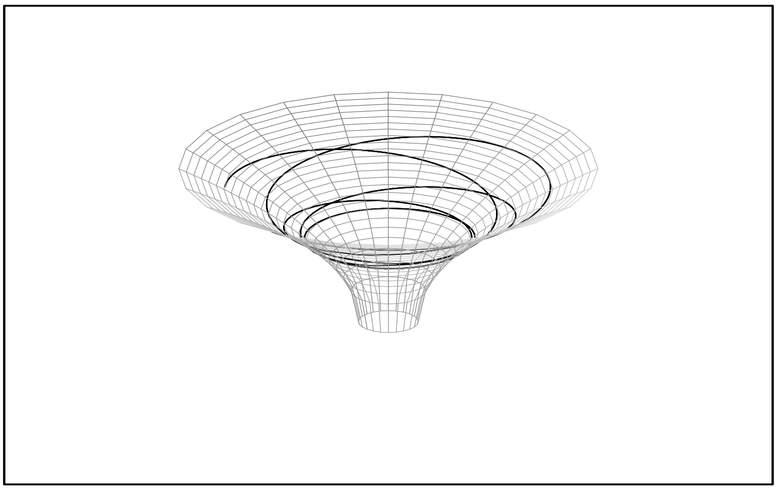

```
> # in plots here and next (for capture, s=0..200) (for unbound

> s=-200..200)
```

This trajectory corresponds to precisely the shortest connection between two points in a curved space. To visualize a curved space, it is convenient to use an embedding diagram; this diagram is a slice of the equatorial plane, with an imaginary depth z to accommodate the actual length. The embedding formula for this problem is

$$z = \sqrt{8M(r - 2M)},$$

We derive this formula in the appendix. For details of an embedding diagram, see [2], p. 615.

```
> p1 := plot3d([x*cos(t), x*sin(t), sqrt(8*(x-2))], x=2..14,
> t=0..2*Pi, style=hidden):  # for capture, make x=2..50
> p2 := spacecurve([rhs(Eq91(s)[3]), rhs(Eq91(s)[2]),
> sqrt(8*(rhs(Eq91(s)[3])-2)), s=0..500],
> coords=cylindrical, color=black, numpoints=400):
> display([p1, p2]);
```



We animate the motion by displaying plot frames in sequence.

```
> N := 50; nstep := 10;
> for i from 0 by nstep to N*nstep do
> pl[i] := polarplot([rhs(Eq91(s)[3]), rhs(Eq91(s)[2]), s=i..i+nstep],
> numpoints=5, color=coral):
> pl3d[i] := spacecurve([rhs(Eq91(s)[3]), rhs(Eq91(s)[2]),
> sqrt(8*(rhs(Eq91(s)[3])-2)), s=i..i+nstep],
> coords=cylindrical, numpoints=5,
> color=black):
> plsum[i] := display(seq(pl[j*nstep], j=0..i/nstep)):
> pl3dsum[i] := display({p1, seq(pl3d[j*nstep], j=0..i/nstep)}):
> end do:
```

```
> display([seq(plsum[i*nstep], i=0..N)], insequence=true,
> scaling=constrained);

> display([seq(pl3dsum[i*nstep], i=0..N)], insequence=true);
```

Maple has a `tensor` package that can calculate geometric properties in curved space and derive the geodesic equations in an integrated command.

6 Conclusion

Maple is a particularly suitable program to handle Lagrangian mechanics, because calculations required for such a theory tend to be tedious even for simple systems. We demonstrate that *Maple* can perform differentiations and simplifications quickly and reliably so that one can avoid lengthy manual (re)arrangement. The equations of motion obtained from the Lagrangian are typically nonlinear differential equations that admit no analytic solution. Under these conditions, numerical solution is necessary. *Maple*'s command `dsolve` spares one a protracted process of programming and debugging with other conventional computer languages. Furthermore, one can generate instant graphics based on numerical solutions, and by experimenting with numbers and observing the output one can develop a profound intuition about the underlying physics.

To illustrate this application of symbolic computation, we select topics over a diverse range: an elementary example of calculus of variations, two problems in advanced classical mechanics, and a problem in general relativity. Traditional physics curriculum places them in various years. Using *Maple*, an instructor can introduce an advanced topic without being restricted to presumed mathematical backgrounds of students, and students can explore more advanced applications without fear of mathematical difficulty. Physics is guided by simple principles; taking advantage of the mathematical power of *Maple*, one can concentrate on physics, instead of technical details of algebraic manipulation.

Appendix

We return to the derivation of an embedding formula for the Schwarzschild geometry. For the equatorial slice, it is a curved two-dimensional space. To visualize the curved space, we introduce an imaginary depth z to accommodate the actual length. At constant t, $\theta(=\pi/2)$, and ϕ, the metric is

$$ds^2 = \frac{dr^2}{1-\dfrac{2M}{r}}. \tag{25}$$

Let us introduce a depth z such that

$$\frac{dr^2}{1-\dfrac{2M}{r}} = dr^2 + dz^2, \tag{26}$$

the depth z is the embedding formula. To solve for z, we need perform only a simple integral.

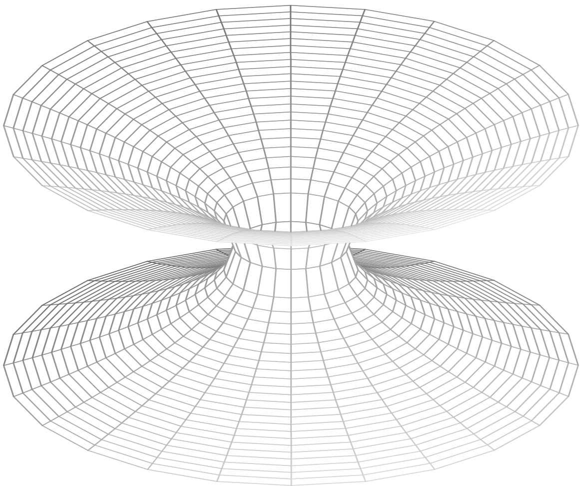
$$z = \int_{2M}^{r} \left(\frac{1}{1-\dfrac{2M}{r'}} - 1 \right)^{1/2} dr' = \sqrt{8M\,(r-2M)} \tag{27}$$

```
> z := int(sqrt(1/(1 - 2*M/u) - 1), u = 2*M..r);
```

$$z := 2\sqrt{2}\sqrt{\frac{M}{r-2M}}\,(r-2M)$$

This simple formula also allows us to make a plot of the Einstein-Rosen bridge, which connects two asymptotically flat universes.

```
> with(plots):

Warning, the name changecoords has been redefined
> p3 := plot3d([r*cos(phi), r*sin(phi), sqrt(8*(r-2))], r=2..10,

> phi=0..2*Pi):
> p4 := plot3d([r*cos(phi), r*sin(phi), -sqrt(8*(r-2))], r=2..10,

> phi=0..2*Pi):
> display([p3, p4], style=hidden);
```



Enthusiasts of science fiction need not be excited by this topology: causality prevents "time travel" using such a solution; for details, again see [2], p. 837.

References

[1] H. Goldstein, *Classical Mechanics*, second edition, Reading, MA: Addison-Wesley, 1980
[2] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, San Francisco, CA: Freeman, 1973